On the Reachability of Linear Time Varying Systems

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Abstract: In this paper system properties of generalized linear time varying (LTV) systems are discussed where, in addition to the control, its certain derivatives also appear both in the dynamics and the observation equation. Developing an adequate version of the Cauchy formula, a necessary and suffiecient condition for complete reachability of generalized LTV systems is obtained in terms of a generalized Gram matrix. Starting from the expansion of coefficient functions in the corresponding Lie algebra basis, we derive an appropriate condition of persistent excitation. The latter leeds to a general condition of complete reachability in terms of quasi-polynomials of the solution of the Wei-Norman equation and differential polynomials of the coefficient functions of the generalized LTV system. Also applying the well-known duality theory of LTV systems, other basic system properties such as controllability, reconstructability and observability can be also treated.

Keywords: Controllability and reachability; Differential algebra; Linear time varying systems; Matrix Lie algebra, persistent excitation

1 Introduction

Definition 1 We call

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \tag{1}$$

a classical linear time varying system in canonic form, where functions

$$A : [0,T] \to \mathbb{R}^{n\times n}, \qquad B : [0,T] \to \mathbb{R}^{n\times k}$$
$$C : [0,T] \to \mathbb{R}^{l\times n}, \qquad D : [0,T] \to \mathbb{R}^{l\times k},$$

are continuous on a fixed interval $[0,T]$.

Remark 1 R. Kalman solved all fundamental problems of such systems (see [9]). He proved the duality both of reachability and observability and of controllability and reconstructability. He also showed the equivalence of reachability-controllability and observability-reconstructability condition pairs for continuous time systems of (1). Therefore we are interested in only one system property, reachability.

Consider the initial value problem

$$\dot{x}(t) = A(t)x(t), \;\; x(\tau) = I \tag{2}$$

in $\mathbb{R}^{n\times n}$. If the coefficient matrix is continuous, then it has a unique solution

$$t \mapsto \Phi(t,\tau) \in \mathbb{R}^{n\times n}$$

defined on the whole interval $[0,T]$, which is continuously differentiable as a two variable function of (t,τ), and $\Phi(t,\tau)$ is invertible for all pairs (t,τ). Consider the solution $t \mapsto \Psi(t,\tau)$ to the initial value problem

$$\dot{Y}(t) = -Y(t)A(t), \quad Y(\tau) = I$$

defined on the whole interval $[0,T]$. Then

$$\begin{aligned}\frac{d}{dt}\big(\Psi(t,\tau)\Phi(t,\tau)\big) &= \dot{\Psi}(t,\tau)\Phi(t,\tau) + \Psi(t,\tau)\dot{\Phi}(t,\tau) = \\ &= \Big(-\Psi(t,\tau)A(t)\Phi(t,\tau) + \Psi(t,\tau)\big(A(t)\Phi(t,\tau)\big)\Big) = 0,\end{aligned}$$

that is, $\Psi(t,\tau)\Phi(t,\tau) = I$ which implies $\Psi(t,\tau) = \Phi\ (t,\tau)^{-1}$ in $\mathbb{R}^n$. Moreover, $\Phi(t,\tau) = \Phi(t,0)\Phi(\tau,0)^{-1}$ because $t \mapsto \Phi(t,0)\Phi(\tau,0)^{-1}$ is a solution to the equation $\dot{x}(t) = A(t)x(t)$ and $\Phi(\tau,0)\Phi(\tau,0)^{-1} = I$. Interchanging t and τ,

$$\Phi(\tau,t) = \Phi(\tau,0)\,\Phi(t,0)^{-1} = \Phi(\tau,0)\,\Psi(t,0),$$

that is,

$$\begin{aligned}\frac{d}{dt}\Phi(\tau,t) &= \Phi(\tau,0)\frac{d}{dt}\Psi(t,0) = \Phi(\tau,0)\big(-\Psi(t,0)A(t)\big) = \\ &= -\Phi(\tau,0)\Phi(t,0)^{-1}A(t) = -\Phi(\tau,t)A(t), \Phi(\tau,\tau) = I.\end{aligned}$$

Therefore

$$\Phi(t,\tau)^{-1} = \Phi(\tau,t).$$

Definition 2 We call

$$G_{\mathrm{Re}a}[0,T] = \int_0^T \Phi(T,t)B(t)B(t)^* \Phi(T,t)^* \, dt$$

the reachability Kalman-Gram matrix.

Theorem 1 (Kalman's reachability theorem) The system (1) is reachable from state 0 if and only if the reachability Kalman-Gram matrix is invertible, or equivalently, positive definite.

Remark 2 A similar theorem holds for controllability. The controllability Kalman-Gram matrix is defined by

$$G_C[0,T] = \int_0^T \Phi(0,t)B(t)B(t)^* \Phi(0,t)^* \, dt.$$

If we define the dual system of (1) as

$$\begin{aligned} \dot{x}(t) &= A(t)^* x(t) + C(t)^* u(t) \\ y(t) &= B(t)^* x(t) + D(t)^* u(t), \end{aligned} \tag{1^*}$$

then the input in (1^*) could be denoted by *y* and the output by *u*, indicating the exchange of their roles.

Theorem 2 (Kalman's duality theorem) (1) is controllable if and only if (1^*) is reconstructable, and (1) is reachable if and only if (1^*) is observable.

Remark 3 Since the dual of the dual system is the original system, (1) is observable if and only if (1^*) is reachable, moreover (1) is reconstructable if and only if (1^*) is controllable.

The observability Kalman-Gram matrix is

$$G_0[0,T] = \int_0^T \Phi(T,t)^* C(t)^* C(t)\Phi(T,t)dt,$$

and reconstructability is equivalent to the invertability (or positive definiteness) of the reconstructability Kalman-Gram matrix

$$G_{\mathrm{Re}}[0,T] = \int_0^T \Phi(0,t)^* C(t)^* C(t)\Phi(0,t)dt.$$

In the following we investigate the reachability of the generalisation of system (1).

Definition 3 A system will be called generalised linear time varying system in canonic form where all functions are assumed to sufficiently smooth:

$$\dot{x}(t) = A(t)x(t) + \sum_{j=0}^{J} B_j(t)u^{(j)}$$
$$y(t) = C(t)x(t) + \sum_{j=0}^{J} D_j(t)u^{(j)}. \qquad (3)$$

Several publications (see for example [4], [6], [7]) deal with the system (1). Based on the above we focus on the general canonic system (3), especially the theoretical construction of the persistent excitation condition, which plays an important role in the existence of a Kalman-like rank condition for linear time-varying systems.

First we recall some basics on Lie algebras.

Definition 4 Let L be a vector space over $\mathbb{R}$ endowed with a multiplication-like operation, the so-called Lie multiplication or Lie bracket: If $l_1, l_2 \in L$ then $[l_1, l_2] \in L$, $l_1 \mapsto [l_1, l_2]$ and $l_2 \mapsto [l_1, l_2]$ are linear mappings and

1) $[l,l]=0$ for all $l \in L$

2) $[l_1,l_2]+[l_2,l_1]=0$ for all $l_1, l_2 \in L$,

3) $[l_1,[l_2,l_3]]+[l_2,[l_3,l_1]]+[l_3,[l_1,l_2]]=0$ for all $l_1,l_2,l_3 \in L$.

endowed with a Lie multiplication is called a Lie algebra.

Remark 4 In the above definition

condition 2 means anticommutativity $[l_1,l_2] = -[l_2,l_1]$, and

condition 3 measures non-associativity.

Indeed,

$$[l_1,[l_2,l_3]] = -[l_2,[l_3,l_1]] - [l_3,[l_1,l_2]] =$$
$$[[l_1,l_2],l_3] - [l_2,[l_3,l_1]],$$

because if $[l_2,[l_3,l_1]] = 0$ then the remaining equation

$$[l_1,[l_2,l_3]] = [[l_1,l_2],l_3]$$

means associativity.

Examples

1) Let $L = \mathbb{R}^{n\times n}$. If we define $[A,B] = AB - BA$ then $\left(\mathbb{R}^{n\times n},[\cdot,\cdot]\right)$ is a Lie algebra.

2) Let $\Omega \subset \mathbb{R}^n$ be an open set and consider the vector space $A(\Omega)$ of analytic functions $f\colon \Omega \to \mathbb{R}^n$. Let the Lie bracket be defined by

$$[f,g](x) = f'(x)g(x) - g'(x)f(x). \tag{4}$$

Then we obtain the Lie algebra $\left(A(\Omega),[\cdot,\cdot]\right)$.

3) Similarly define the real vector space of infinite times differentiable functions $C^{\infty}(\Omega)$ on an open set $\Omega \subset \mathbb{R}^n$ and define the Lie multiplication as in (4). Then we obtain the Lie algebra $\left(C^{\infty}(\Omega),[\cdot,\cdot]\right)$.

Definition 5 Consider system (3). The sub-Lie-algebra $L \subset \mathbb{R}^{n\times n}, \left(L,[\cdot,\cdot]\right)$ generated by

$$\left\{A(t):\ t \in [0,T]\right\} \subset \mathbb{R}^{n\times n}$$

is defined as the smallest Lie-algebra for which $\left\{A(t):\ t \in [0,T]\right\} \subset L$ holds.

Remarks 5

Such a sub-Lie-algebra exists because the set of the containing sub-Lie-algebras is non-empty, $\mathbb{R}^{n\times n}$ is an element, and the intersection of these is the minimal sub-Lie-algebra generated by $\left\{A(t)\right\}$.

Since $\mathbb{R}^{n\times n}$ has finite dimension (n^2), the sub-Lie-algebra $L \subset \mathbb{R}^{n\times n}$ is also finite dimensional.

Let $A_1, A_2, \ldots, A_I \in L$ be a basis of L. In this basis

$$A(t) = \sum_{i=1}^{I} a_i(t) A_i,$$

and the Lie bracket $[A_i, A_j] \in L$

$$[A_i, A_j] = \sum_{k=1}^{I} \Gamma_{ij}^{k} A_k.$$

Since $X \mapsto [A_i, X] = AdA_i(X)$ is a linear mapping on the vector space L (which is also a Lie algebra), the matrix representation of AdA_i in the basis $A_1, A_2, \ldots, A_I$ can be expressed with the help of the numbers Γ_{ij}^{k}.

Let $X = \sum_{j=1}^{I} x_j A_j$. Then

$$[A_i, X] = \left[A_i, \sum_{j=1}^{I} x_j A_j\right] = \sum_j x_j \left(\sum_{h=1}^{I} \Gamma_{ij}^{h} A_h\right),$$

$\sum_j \Gamma_{ij}^h x_j$ is the h^{th} component of the matrix-vector product

$$\begin{pmatrix} \Gamma_{i1}^1 & \Gamma_{i2}^1 & \dots & \Gamma_{iI}^1 \\ \Gamma_{i1}^2 & \Gamma_{i2}^2 & \dots & \Gamma_{iI}^2 \\ \vdots & \vdots & \vdots & \vdots \\ \Gamma_{i1}^I & \Gamma_{i2}^I & \dots & \Gamma_{iI}^I \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_I \end{pmatrix} = \Gamma_i \mathbf{x}$$

and because of the the correspondences $AdA_i X \leftrightarrow \Gamma_i \mathbf{x},\ X \leftrightarrow \mathbf{x}$, Γ_i is the representation of the matrix AdA_i in the basis $A_1, A_2, ..., A_l \in L$.

According to the Cauchy formula the solution to system with initial condition $x(0) = \xi$ is

$$x(t) = \Phi(t,0)\xi + \int_0^t \Phi(t,\tau)B(\tau)u(\tau)d\tau,$$

and similarly in the case of the generalized system (3) in canonic form:

$$x(t) = \Phi(t,0)\xi + \int_0^t \Phi(t,\tau)\left(\sum_j B_j(\tau)u^{(j)}(\tau)\right)d\tau.$$

For systems with constant coefficients the basic solutions $\Phi(t,\tau)$ are the solutions to

$$\dot{x}(t) = Ax(t), \qquad x(\tau) = I$$

i.e.,

$$\Phi(t,\tau) = \exp A(t-\tau).$$

Moreover, if the system's structure matrix $A(t)$ has the form $A(t) = a(t)A$ then the basic solutions are $\Phi(t,\tau) = \exp A\int_\tau^t a(s)ds$.

In the case of system (3)

$$\dot{x}(t) = \sum_{i=1} a_i(t)A_i x + \sum_j B_j(t)u^{(j)}(t)$$

the basic solutions take the form

$$\Phi(t,\tau) = \exp A_1 g_1(t,\tau) \exp A_2 g_2(t,\tau) \dots \exp A_I g_I(t,\tau).$$

Again, we assume that $A_1, A_2, \ldots, A_I$ is a basis in the Lie algebra L generated by $A(t)$ and also that matrix $\Gamma_i \in \mathbb{R}^{IxI}$ is the representation of AdA_i. Then the existence of the above representation is guaranteed by the Wei-Norman theorem.

Wei-Norman Theorem

Let $\gamma(t) = g(t,\tau) \in \mathbb{R}^k$ be a solution to the so-called Wei-Norman nonlinear differential equation

$$\left(\sum_{i=1}^{I} \left(\exp\Gamma_1\gamma_1 \exp\Gamma_2\gamma_2 \ldots \exp\Gamma_{i-1}\gamma_{i-1} \right) \cdot E_{ii} \right) \dot{\gamma} = \mathbf{a} \qquad (5)$$
$$\gamma(\tau) = 0$$

where vector **a** is composed from functions a_i and E_{ii} is the (0,-1) matrix with single 1 entry at the i^{th} diagonal element.

Then

$$\Phi(t,\tau) = \exp A_1 g(t,\tau) \exp A_2 g(t,\tau) \ldots \exp A_I g(t,\tau).$$

Remarks 6 The solution locally exists, because the initial condition $\gamma(\tau) = 0$ implies that the matrix to be inverted at $\tau = 0$ is the identity, which is invertible, thus also invertible in an appropriate neighbourhood of τ, and so can be made explicit.

It is well-known (cf. [3]) that

$$\exp A_i g_i(t,\tau) = \sum_{j=0}^{n-1} q_{ij} \left(g_i(t,\tau) \right) A_i^j \qquad (6)$$

is a polynomial of A_i with maximal degree $n-1$, a quasipolynomial of $g_i(t,\tau)$ that is, a polynomial of $g_i(t,\tau)$, $\sin\alpha_l g_i(t,\tau)$, $\cos\beta_l g_i(t,\tau)$ and exponential of $\lambda_l g_i(t,\tau)$, where λ_l are the real parts and α_l, β_l are the imaginary parts of the eigenvalues of A_i. (The basic results can be found in the classical monographs of matrix theory and ordinary differential equations, such as [3]).

Substituting (6) into the exponential product we obtain

$$\Phi(t,\tau) = \sum_{\mathbf{n}} Q_{\mathbf{n}} \left(\mathbf{g}(t,\tau) \right) A_1^{n_1} A_2^{n_2} \ldots A_I^{n_I},$$

where $Q_{\mathbf{n}}$ is a quasipolynomial of $\mathbf{g}(t,\tau) = \left(g_1(t,\tau), g_2(t,\tau), \ldots, g_I(t,\tau) \right)$ (a certain product of quasipolynomials $g_{ij}\left(g_i(t,\tau) \right)$).

2 Reachability of the Canonic System

Lemma 1 Consider the final states of the generalized linear system with time-varying coefficients (3) and with the initial condition $x(0)=0$. Then with appropriate integer constants $C_{\alpha,\beta}$

$$x(T)=\int_0^T \Phi(T,t)\left[\sum_{j=0}^{J}\left(\sum_{|\boldsymbol{\alpha}+\boldsymbol{\beta}|=j-\hat{j}} C_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left(A^{(\boldsymbol{\alpha})}(t)\right)^{\boldsymbol{\beta}}\right)B_j^{(\hat{j})}(t)\right]u(t)dt.$$

Here we apply the following notation:

for integers $0\le\alpha_1,\alpha_2,\dots,\alpha_\gamma$, $1\le\beta_1,\beta_2,\dots,\beta_\gamma$,

$$\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\dots,\alpha_\gamma),\quad \boldsymbol{\beta}=(\beta_1,\beta_2,\dots,\beta_\gamma),$$

$$|x|=\sum|\alpha|,$$

$$\left(A^{(\boldsymbol{\alpha})}(t)\right)^{\boldsymbol{\beta}}=\left(A^{(\alpha_1)}(t)\right)^{\beta_1}\left(A^{(\alpha_2)}(t)\right)^{\beta_2}\dots\left(A^{(\alpha_\gamma)}(t)\right)^{\beta_\gamma}.$$

Proof By the Cauchy formula,

$$x(T)=\int_0^T \Phi(T,t)\left(\sum_j B_j(t)u^{(j)}(t)\right)dt. \tag{7}$$

If the highest order derivative of u that appears is $u^{(J)}(t)$ then we assume that for all $j=0,1,2,\dots,J-1$ the boundary conditions $u^{(j)}(0)=0$, $u^{(j)}(T)=0$ hold. This assumption does not affect the subspace of the reachable final states in $\mathbb{R}^n$.

$$\begin{aligned}x(T)&=\int_0^T\Phi(T,t)B_0(t)u(t)dt+\sum_{j=1}\int_0^T\Phi(T,t)B_j(t)u^{(j)}(t)dt=\\&=\int_0^T\Phi(T,t)B_0u(t)dt+\sum_{j\ge1}[\Phi(T,t)B_j(t)u^{(j-1)}(t)]_0^T-\\&-\sum_{j=1}\int_0^T\frac{d}{dt}\left(\Phi(T,t)B_j(t)\right)u^{(j-1)}(t)dt=\int_0^T\Phi(T,t)B_0(t)u(t)dt+\\&+\int_0^T\Phi(T,t)\left(A(t)B_1(t)-B_1'(t)\right)u(t)dt+\sum_{j\ge2}\int_0^T\Phi(T,t)\Big(A(t)B_j(t)-B_j'(t)\Big)u^{(j-1)}(t)dt.\end{aligned}$$

Repeating this for the last term we obtain the equations of the next step:

$$\sum_{j\geq 2}\int_0^T \Phi(T,t)\Big(A(t)B_j(t)-B_j'(t)\Big)u^{(j-1)}(t)dt=\sum_{j\geq 2}\Big[\Phi(T,t)\big(A(t)B_j(t)-B_j'(t)\big)u^{(j-2)}(t)\Big]_0^T-$$

$$-\int_0^T \frac{d}{dt}\Big(\Phi(T,t)\big(A(t)B_2(t)-B_2'(t)\big)\Big)u(t)dt-\sum_{j\geq 3}\int_0^T \frac{d}{dt}\Big(\Phi(T,t)\big(A(t)B_j(t)-B_j'(t)\big)\Big)$$

$$u^{(j-2)}(t)dt=\int_0^T \Phi(T,t)\big(A(t)^2B_2(t)-2A(t)B_2'(t)-A'(t)B_2(t)+B''(t)\big)u(t)dt+$$

$$+\sum_{j\geq 3}\int_0^T \Phi(T,t)\big(A(t)^2B_j(t)-2A(t)B_j'(t)-A'(t)B_j(t)+B_j''(t)\big)u^{(j-2)}(t)dt.$$

Again, integrating by parts in the last term, we have a similar equation:

$$\sum_{j\geq 3}\int_0^T \Phi(T,t)\Big(A(t)^2B_j(t)-2A(t)B_j'(t)-A'(t)B_j(t)+B_j''(t)\Big)u^{(j-2)}(t)dt=$$

$$=\int_0^T \Phi(T,t)\Big(A(t)^3B_3(t)-2A(t)A(t)'B_3(t)-A'(t)A(t)B_3(t)-3A(t)^2B_3'(t)+3A'(t)B_3'(t)+$$

$$+3A(t)B_3''(t)-B_3'''(t)\Big)u(t)dt+$$

$$+\sum_{j\geq 4}\int_0^T \Phi(T,t)\Big(A(t)^3B_j(t)-3A(t)^2B_j'(t)-2A(t)A'((t))B_j(t)+$$

$$+3A(t)B_j''(t)-A'(t)A(t)B_j(t)+3A'(t)B_j'(t)+A''(t)B_j(t)-B_j'''(t)\Big)u^{(j-3)}(t)dt.$$

After the j^{th} step, no derivative of $u(t)$ appears in the integral. Then (applying the above notation)

$$x(T)=\int_0^T \Phi(T,t)B_0(t)u(T)dt+\int_0^T \Phi(T,t)\Big(A(t)B_1(t)-B_1'(t)\Big)u(t)dt+$$

$$+\int_0^T \Phi(T,t)\Big(A(t)^2-A'(t)B_2(t)-2A(t)B_2'(t)+B_2''(t)\Big)u(t)dt+$$

$$+\int_0^T \Phi(T,t)\Big(\big(A(t)^3-2A(t)A'(t)-A'(t)A(t)\big)B_3(t)-$$

$$-\big(3A(t)^2-3A'(t)\big)B_3'(t)+3A(t)B_3''-B_3'''\Big)u(t)dt+$$

$$+\ldots=$$

$$=\int_0^T \Phi(T,t)\left[\sum_{j=0}^{J}\left(\sum_{|\alpha+\beta|=j-\hat{j}} C_{\alpha,\beta}\left(A^{(\alpha)}(t)\right)^{\beta}\right)B_j^{(\hat{j})}(t)\right]u(t)dt,$$

which was to be proved. ■

Definition 8 The reachability Kalman-Gram matrix for generalised systems is defined as

$$G_{\mathrm{Re}\,a}[0,T] =$$
$$\int_0^T \Phi(T,t)\left[\sum_{j=0}^{J}\left(\sum_{|\boldsymbol{\alpha}+\boldsymbol{\beta}|=j-\hat{j}} C_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left(A^{(\alpha)}(t)\right)^{\boldsymbol{\beta}}\right)B_j^{(\hat{j})}(t)\right]\cdot$$
$$\cdot\left[\sum_{j=0}^{J}\left(\sum_{|\boldsymbol{\alpha}+\boldsymbol{\beta}|=j-\hat{j}} C_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left(A^{(\alpha)}(t)\right)^{\boldsymbol{\beta}}\right)B_j^{(\hat{j})}(t)\right]^{Tr} \Phi(T,t)^{Tr}\,dt.$$

Theorem 3 *The general linear time-varying system on the interval* $[0,T]$ *is completely reachable if and only if the Kalman-Gram matrix* $G_{\mathrm{Re}\,a}[0,T]$ *is invertible, or equivalently, positive definite.*

Proof Based on our first lemma (Lemma 1) the proof is similar to the case of classical time-varying linear systems.

Starting from the expansion of coefficient functions in the corresponding Lie algebra basis, we derive an appropriate condition of persistent excitation. The latter leeds to a general condition of complete reachability in terms of quasi-polynomials of the solution of the Wei-Norman equation and differential polynomials of the coefficient functions of the generalized LTV system. Also applying the well-known duality theory of LTV systems, other basic system properties such as controllability, reconstructability and observability can be also treated.

Let L be the Lie algebra generated by $\{A(t):\ t\in[0,T]\}\subset\mathbb{R}^{n\times n}$ and let $A_1, A_2,\dots, A_I \in L$ be a basis. In this basis

$$A(t)=\sum_{i=1}^{I} a_i(t)A_i.$$

Similarly, for the matrices $B_0(t), B_1(t),\dots, B_J(t)$, if

$$V = V\left\{B_0(t), B_1(t),\dots, B_J(t):\ t\in[0,T]\right\}\subset R^{n\times k}$$

is the subspace of the vector space $\mathbb{R}^{n\times k}$ spanned by $B_j(t)$ then a basis $B_1, B_2,\dots, B_{\hat{I}}$ can be chosen in V such that

$$B_j(t)=\sum_{\hat{i}=1}^{\hat{I}} b_{j\hat{i}}(t)B_{\hat{i}}.$$

Now $\Phi(T,t)$ can be written as a polynomial of $A_1, A_2,\dots, A_I$, a quasi-polynomial of the solutions $g_1, g_2,\dots, g_I$ to the Wei-Norman equation [8], and the kernel function $\sum_{j=0}^{J}\left(\sum_{|\boldsymbol{\alpha}+\boldsymbol{\beta}|=j-\hat{j}} C_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left(A^{(\boldsymbol{\alpha})}(t)\right)^{\boldsymbol{\beta}}\right)B_j^{(\hat{j})}(t)$ in the integral form of $x(T)$ can be written as a polynomial of $A_1, A_2, \dots, A_I$ and $B_1, B_2, \dots, B_{\hat{I}}$ and a differential polynomial of $a_i(t)$ and $b_{j\hat{i}}(t)$ (with integer coefficients and first degree $B_{\hat{i}}$). Exchanging adjacent terms, the powers of A_i can be arranged in the natural order (having the form $A_1^{m_1}, A_2^{m_2}, \dots, A_I^{m_I}$, where all m_i satisfy $0 \le m_i \le n_i$, or $\mathbf{0} \le \mathbf{m} < \mathbf{n}$) using the equations $A_{i_1}A_{i_2} = A_{i_2}A_{i_1} + \sum_{h=1}^{I}\Gamma_{i_1 i_2}A_h$ and $A_i^n = \sum_{\hat{i}=0} C_{i\hat{i}} A_i^{\hat{i}}$ assuming that the characteristic polynomial of A_i has the form $\lambda^n = \sum_{i=0}^{n-1} C_{i\hat{i}}\lambda^i$.

Then we have

$$x(T) = \sum_{\mathbf{0}\le\mathbf{m}<\mathbf{n}}\sum_{\hat{i}=0}^{\hat{I}} A_1^{m_1}\cdot A_2^{m_2}\cdot\ldots\cdot A_I^{m_I} B_{\hat{i}}\int_0^T P_{\mathbf{m},\hat{i}}$$
$$\left(\mathbf{g}(T,t), \mathbf{a}^{[\infty)}(t), \mathbf{b}_1^{[\infty)}(t), \dots, \mathbf{b}_J^{[\infty)}(t)\right)u(t)dt.$$

Here $P_{\mathbf{m},\hat{i}}\left(\mathbf{g}(T,t), \mathbf{a}^{[\infty)}(t), \dots, b_j^{[\infty)}(t), \dots\right)$ are quasi-polynomials of $g_1(T,t), g_2(T,t), \dots, g_I(T,t)$ and differential polynomials of $a_1(t), \dots, a_I(t)$, and $b_{11}(t), b_{12}(t), \dots, b_{1\hat{I}}(t)$, $b_{21}(t), \dots, b_{2\hat{I}}(t), \dots, b_{J1}(t), b_{J2}(t), \dots, b_{J\hat{I}}(t)$, and the following notation is used

$$\mathbf{a}^{[\infty)} = (a, a', a'', \dots, a^{(\mu)}),$$
$$\mathbf{b}^{[\infty)} = (b, b', b'', \dots, b^{(\mu)}),$$

where μ, ν are arbitrary (but finite) nonnegative integers.

This implies that the reachable subspace of the general system on [0,T] must be a subset of the image space

$$\operatorname{Im}\left\{\dots, A_1^{m_1}\cdot A_2^{m_2}\cdot\ldots\cdot A_I^{m_I}\cdot B_{\hat{i}}, \dots\right\}$$

which is similar to the case of classical canonical systems.

The reachability subspace is extended because the derivatives can also be inputs, therefore let

$$V_0 = V\left\{B_0(t);\ t\in[0,T]\right\} \subset \mathbb{R}^{n\times k},$$

$$V_J = V\left\{B_0(t), B_1(t), \dots, B_J(t);\ t\in[0,T]\right\} \subset \mathbb{R}^{n\times k}$$

be the subspaces generated by the corresponding $B_J(t)$ matrices. Choose a basis of V_J such that the first $\hat{I}_0$ elements form a basis of V_0:

$$V_0 = V\left\{B_1, B_2, \ldots, B_{\hat{I}_0}\right\},$$
$$V_J = V\left\{B_1, B_2, \ldots, B_{\hat{I}_0},\ B_{\hat{I}_0+1}, B_{\hat{I}_0+2}, \ldots, B_{\hat{I}_J}\right\}$$

From this it is obvious that for the general system, the image of the corresponding generalized Kalman-matrices (briefly Kalman matrices in the following) contains the image of the general Kalman matrices of the classical system.

From the proofs for the classical system one can deduce the persistent excitation condition which guarantees that the reachability subspace of the general system coincides with the image of the image of the general Kalman-matrix of the system.

Suppose that ξ is a vector in the image space of the Kalman-matrix,

$$K_{\text{gen}} = \left\{\ldots, A_1^{m_1} \cdot A_2^{m_2} \cdot \ldots \cdot A_I^{m_I} \cdot B_{\hat{i}}, \ldots\right\}.$$

K_{gen} does not equal the reachable subspace of the general system over $[0,T]$ if and only if there exists $\xi \neq 0$, $\xi \in \operatorname{Im}(K_{\text{gen}})$, such that $\langle \xi, x(T)\rangle = 0$ for all possible inputs $u(t)$.

This means

$$0 = \langle \xi, x(T)\rangle =$$
$$= \left\langle \xi, \sum_{\mathbf{0}\le\mathbf{m}<\mathbf{n}} \sum_{\hat{i}=0}^{\hat{I}} A_1^{m_1} A_2^{m_2} \ldots A_I^{m_I} B_{\hat{i}} \int_0^T P_{\mathbf{m},\hat{i}}\left(\mathbf{g}(T,t), \mathbf{a}^{[\infty)}(t), \ldots, \mathbf{b}_j^{[\infty)}, \ldots\right) u(t)\,dt \right\rangle =$$
$$= \int_0^T \left\langle \sum_{\mathbf{0}\le\mathbf{m}<\mathbf{n}} \sum_{\hat{i}=0}^{\hat{I}} P_{\mathbf{m},\hat{i}}\left(\mathbf{g}(T,t)\,\mathbf{a}^{[\infty)}(t)\ldots \mathbf{b}_j^{[\infty)} \ldots\right)\left(A_I^T\right)^{m_I}\left(A_{I-1}^T\right)^{m_{I-1}} \ldots \left(A_1^T\right)^{m_1} B_{\hat{i}}^T \xi, u(t)\right\rangle dt.$$

By the classical Lagrange lemma, if the above holds for every "nice", e.g. continuous function u then

$$\sum_{\mathbf{0}\le\mathbf{m}<\mathbf{n}} \sum_{\hat{i}=0}^{\hat{I}} P_{\mathbf{m},\hat{i}}\left(g(T,t), \mathbf{a}^{[\infty)}(t), \ldots, \mathbf{b}_j^{[\infty)}, \ldots\right) \left(A_I^T\right)^{m_I}\left(A_{I-1}^T\right)^{m_{I-1}} \ldots \left(A_1^T\right)^{m_1} B_{\hat{i}}^T \xi = 0 \tag{8}$$

The analytic functions $\exp\lambda g, \cos\alpha g, \sin\alpha g$ in the quasi-polynomials $P_{\mathbf{m},\hat{i}}$ can be replaced by new variables $\overline{g} = \exp\lambda g$, $\hat{g} = \cos\alpha g$, $\breve{g} = \sin\alpha g$. The corresponding differential equations are:

$$\begin{aligned} \dot{\overline{g}} &= \lambda \dot{g} \exp\lambda g = \lambda \dot{g}\overline{g}, \quad \text{and} \\ \dot{\hat{g}} &= -\alpha \dot{g} \sin\alpha g = -\alpha \dot{g}\breve{g}, \\ \dot{\breve{g}} &= \alpha \dot{g} \cos\alpha g = \alpha \dot{g}\hat{g}. \end{aligned} \tag{9}$$

To make these differential equations explicit consider the Wei-Norman differential equation

$$\left(\sum_{i=1}^{I} \exp\Gamma_1 g_1 \exp\Gamma_2 g_2 \ldots \exp\Gamma_{i-1} g_{i-1} E_{ii} \right) \dot{\mathbf{g}} = \mathbf{a},\ \mathbf{g}(0) = \mathbf{0}.$$

The exponential products are exponents in the multiplication table of the Lie algebra $\Gamma_1, \Gamma_2, \ldots, \Gamma_{I-1}$ (the Christoffel symbols). Again, we can introduce the non-polynomial terms $\exp\lambda g, \cos\alpha g, \sin\alpha g$ as new variables, which means adding more differential equations of the type (9) that are polynomial. Thus the Wei-Norman equation becomes polynomial but non-explicit.

The equation can be made explicit in the original derivatives $\dot{g}$:

$$\dot{g} = \left(\sum_{i=1}^{I} \exp\Gamma_1 g_1 \exp\Gamma_2 g_2 \ldots \exp\Gamma_{i-1} g_{i-1} E_{ii} \right)^{-1} \mathbf{a},$$

and equations (9) also become explicit with fractional denominators:

$$\det\left(\sum_{i=1}^{I} \exp\Gamma_1 g_1 \exp\Gamma_2 g_2 \ldots \exp\Gamma_{i-1} g_{i-1} E_{ii} \right).$$

Multiplying the system of explicit equations by these, in the end we obtain an implicit polynomial differential equation with variables $\mathbf{g}, \overline{\mathbf{g}}, \overset{\frown}{\mathbf{g}}, \mathbf{g}$, where each equation contains only one derivative, that is, a regular differential equation which can be made explicit in the derivatives (with fractional right hand sides).

Thus the quasi-polynomials $P_{\mathbf{m},\hat{i}}\left(\mathbf{g}(T,t), \mathbf{a}^{[\infty)}(t), \mathbf{b}_1^{[\infty)}(t), \mathbf{b}_2^{[\infty)}(t), \ldots, \mathbf{b}_I^{[\infty)}(t)\right)$ can be replaced by polynomials

$$\overline{P}_{\mathbf{m},i}\left(\mathbf{g}(T,t), \overline{\mathbf{g}}(T,t), \hat{g}(T,t), \breve{g}(T,t), \mathbf{a}^{[\infty)}(t), \mathbf{b}_1^{[\infty)}(t), \mathbf{b}_2^{[\infty)}(t), \ldots, \mathbf{b}_I^{[\infty)}(t)\right)$$

of variables $\mathbf{g}, \overline{\mathbf{g}}, \hat{\mathbf{g}}, \breve{\mathbf{g}}$ and differential polynomials of functions $\mathbf{a}(t)$ $\mathbf{b}_0(t), \mathbf{b}_1(t), \ldots, \mathbf{b}_J(t)$.

Denote the variables $\mathbf{g},\overline{\mathbf{g}},\hat{\mathbf{g}},\breve{\mathbf{g}}$ by $\mathbf{x}=(x_1,x_2,\ldots,x_N)$ and by $\mathbf{u}=(u_1,u_2,\ldots,u_K)$, and rewrite the above implicit polynomial differential equation as

$$F(\mathbf{x},\dot{\mathbf{x}},\mathbf{u},\ddot{\mathbf{u}},\ldots)=0.$$

We also rewrite equation (8) using $\mathbf{x},\mathbf{u}$

$$\sum_{\mathbf{0}\le\mathbf{m}<\mathbf{n}}\sum_{\hat{i}=0}^{I}\overline{P}_{bm,\hat{i}}(\mathbf{x},\dot{\mathbf{x}},\mathbf{u},\dot{\mathbf{u}},\ddot{\mathbf{u}},\ldots)\left(A_I^T\right)^{m_I}\left(A_{I-1}^T\right)^{m_{I-1}}\ldots\left(A_1^T\right)^{m_1}B_{\hat{i}}^T\xi=0.$$

Define the output equation

$$y=\sum_{\mathbf{0}\le\mathbf{m}<\mathbf{n}}\sum_{\hat{i}=0}^{I}\overline{P}_{\mathbf{m},\hat{\mathbf{i}}}(\mathbf{x},\mathbf{u},\dot{\mathbf{u}},\ldots,)\left(A_I^T\right)^{m_I}\left(A_{I-1}^T\right)^{m_{I-1}}\ldots\left(A_1^T\right)^{m_1}B_{\hat{i}}^T\xi=G\left(\mathbf{x},u,\dot{u},\ldots,\xi\right).$$

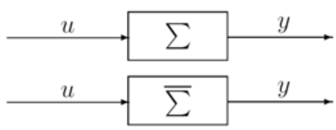
Thus we have an input-output system

$$\begin{aligned}F\left(\mathbf{x},\dot{\mathbf{x}},\mathbf{u},\dot{\mathbf{u}},,\ldots,\xi\right)&=0\\ \mathbf{y}=G(\mathbf{x},\mathbf{u},\dot{\mathbf{u}},,\ldots,\xi),&\end{aligned}\qquad(\Sigma)$$

which is polynomial and implicit in the derivatives $\dot{x}$, with the regularity condition $\partial_{\dot{x}}F(\mathbf{x},\dot{\mathbf{x}},\mathbf{u},\dot{\mathbf{u}},\ddot{\mathrm{u}},\ldots,\xi)\neq 0$. Here **u** are the inputs, **x** are the states and **y** are the outputs. Consider another representation with possibly different states but with the same inputs and outputs

$$\begin{aligned}\overline{F}\left(\overline{\mathbf{x}},\dot{\overline{\mathbf{x}}},\mathbf{u},\dot{\mathbf{u}},,\ldots,\xi\right)&=0\\ y=\overline{G}(\overline{\mathbf{x}},\mathbf{u},\dot{\mathbf{u}},,\ldots,\xi).&\end{aligned}\qquad(\overline{\Sigma})$$

Let $(\sum)$ and $(\overline{\sum})$ be input-output systems. We call them equivalent if for every input-output pair (u,y), has a solution $\mathbf{x}$ if and only if has a solution $\overline{\mathbf{x}}$. In this case the two systems



behave in the same way. $\sum$ and $\overline{\sum}$ can be written more briefly also allowing the derivatives of the outputs y

$$J(\mathbf{x},\dot{\mathbf{x}},\mathbf{u},\dot{\mathbf{u}},\ddot{\mathbf{u}},\ldots,\mathbf{y},\dot{\mathbf{y}},\ddot{\mathbf{y}},\ldots,\xi)=0,\qquad(10)$$

and

$$\overline{J}(\mathbf{x},\dot{\mathbf{x}},\mathbf{u},\dot{\mathbf{u}},\ddot{\mathbf{u}},\ldots,\mathbf{y},\dot{\mathbf{y}},\ddot{\mathbf{y}},\ldots,\xi)=0.$$

Diop [1] proved the existence of a finite purely algebraic algorithm which gives differential polynomials

$$\hat{J}(\mathbf{u},\dot{\mathbf{u}},\ddot{\mathbf{u}},\ldots,\mathbf{y},\dot{\mathbf{y}},\ddot{\mathbf{y}},\ldots,\xi)$$
$$\hat{G}(\mathbf{u},\dot{\mathbf{u}},\ddot{\mathbf{u}},\ldots,\mathbf{y},\dot{\mathbf{y}},\ddot{\mathbf{y}},\ldots,\xi)$$

such that the system (10) is equivalent to the input-output system $\mathbf{u} \mapsto \dot{\mathbf{y}}$ defined by the implicit equation and the non-equality condition

$$\begin{aligned}\hat{J}(\mathbf{u},\dot{\mathbf{u}},\ddot{\mathbf{u}},\ldots,\mathbf{y},\dot{\mathbf{y}},\ddot{\mathbf{y}},\ldots,\xi) = 0\\ \hat{G}(\mathbf{u},\dot{\mathbf{u}},\ddot{\mathbf{u}},\ldots,\mathbf{y},\dot{\mathbf{y}},\ddot{\mathbf{y}},\ldots,\xi) \neq 0.\end{aligned} \tag{11}$$

The latter has no state variable $\mathbf{x}$ thus we can call the Diop algorithm a state elimination algorithm.

Definition 9 (10) and (11) define equivalent input-output systems if for any input-output pair (u, y), (10) has a solution with respect to the state $\mathbf{x}$, i.e. the triple (x,u,y) is a solution to (10) if (u, y) is a solution to the polynomial equation

$$\hat{J}(\mathbf{u},\dot{\mathbf{u}},\ddot{\mathbf{u}},\ldots,\mathbf{y},\dot{\mathbf{y}},\ddot{\mathbf{y}},\ldots,\xi) = 0$$

and

$$\hat{G}(\mathbf{u},\dot{\mathbf{u}},\ddot{\mathbf{u}},\ldots,\mathbf{y},\dot{\mathbf{y}},\ddot{\mathbf{y}},\ldots,\xi) \neq 0$$

holds.

Remark 7 We get this latter result by dividing by a differential polynomial in each step of the algorithm, and since the divisor obviously cannot be *0*, this must be assumed. Their products form the differential polynomials $\hat{G}(\mathbf{u},\dot{\mathbf{u}},\ddot{\mathbf{u}},\ldots,\mathbf{y},\dot{\mathbf{y}},\ddot{\mathbf{y}},\ldots,\xi)$. If this product $\neq 0$ then neither of its factors can be 0.

Now returning to the original input-output system $(\sum)$, taking the "input" $\mathbf{u} = (\mathbf{a},\mathbf{b}_0,\mathbf{b}_1,\ldots,\mathbf{b}_J)$ ordered continuously in a row, we have

$$F\Big(\mathbf{x},\dot{\mathbf{x}},(\mathbf{a},\mathbf{b}_0,\mathbf{b}_1,\ldots,\mathbf{b}_j),(\dot{\mathbf{a}},\dot{\mathbf{b}}_0,\dot{\mathbf{b}}_1,\ldots,\dot{\mathbf{b}}_j),\ldots,\xi\Big) = 0,$$
$$G\Big(\mathbf{x},(\mathbf{a},\mathbf{b}_0,\mathbf{b}_1,\ldots,\mathbf{b}_j),(\dot{\mathbf{a}},\dot{\mathbf{b}}_0,\dot{\mathbf{b}}_1,\ldots,\dot{\mathbf{b}}_j),\ldots,\xi\Big) = 0.$$

Substituting the input-output pair $\big((\mathbf{a},\mathbf{b}_0,\mathbf{b}_1,\ldots,\mathbf{b}_J),0\big)$ into the state eliminated system obtained from system $(\sum)$, the equivalence of the systems yields the equation and non-equality

$$\begin{aligned}&\hat{J}\left((\mathbf{a},\mathbf{b}_0,\mathbf{b}_1,\ldots,\mathbf{b}_J),(\dot{\mathbf{a}},\dot{\mathbf{b}}_0,\dot{\mathbf{b}}_1,\ldots,\dot{\mathbf{b}}_J),\ldots,0,0,0,\ldots,\xi\right)=0\\&\hat{G}\left((\mathbf{a},\mathbf{b}_0,\mathbf{b}_1,\ldots,\mathbf{b}_J),(\dot{\mathbf{a}},\dot{\mathbf{b}}_0,\dot{\mathbf{b}}_1,\ldots,\dot{\mathbf{b}}_J),\ldots,0,0,0,\ldots,\xi\right)\neq 0\end{aligned} \tag{12}$$

which give a sufficient condition that for all inputs $\mathbf{u}$, the end state $x(T)$ is orthogonal to the given vector

$$\xi\in\operatorname{Im}\left\{\ldots,A_1^{m_1}A_2^{m_2}\ldots A_I^{m_I}B_j,\ldots\right\}. \tag{13}$$

Definition 10 We say that the time-variant coefficients $\mathbf{a},\mathbf{b}_0,\mathbf{b}_1,\ldots,\mathbf{b}_J$ persistently excite the system if the subspace of the reachable states coincides with the image space of the generalized Kalman-matrix, i.e., it is the largest possible subspace. According to our equations, if the coefficients satisfy the conditions (12) then state ξ must be 0.

The most interesting special case is when the image space of the generalized Kalman-matrix is the whole space $\mathbb{R}^n$. Then the coefficients $\mathbf{a},\mathbf{b}_0,\mathbf{b}_1,\ldots,\mathbf{b}_J$ persistently stimulate the system if and only if the system is totally reachable on the interval $[0,T]$.

Thus if the coefficients do not persistently excite the system then

$$\begin{gathered}\hat{J}\left((\mathbf{a},\mathbf{b}_0,\mathbf{b}_1,\ldots,\mathbf{b}_J),(\dot{\mathbf{a}},\dot{\mathbf{b}}_0,\dot{\mathbf{b}}_1,\ldots,\dot{\mathbf{b}}_J),\ldots,0,0,0,0,\ldots,\xi\right)=0\\\hat{G}\left((\mathbf{a},\mathbf{b}_0,\mathbf{b}_1,\ldots,\mathbf{b}_J),(\dot{\mathbf{a}},\dot{\mathbf{b}}_0,\dot{\mathbf{b}}_1,\ldots,\dot{\mathbf{b}}_J),\ldots,0,0,0,0,\ldots,\xi\right)\neq 0\\\xi\neq 0\end{gathered}$$

can be solved. Regarding the equation as an implicit function of ξ, it can be solved for ξ,

$$\boldsymbol{\xi}=\hat{f}\left((\mathbf{a},\mathbf{b}_0,\mathbf{b}_1,\ldots,\mathbf{b}_J),(\dot{\mathbf{a}},\dot{\mathbf{b}}_0,\dot{\mathbf{b}}_1,\ldots,\dot{\mathbf{b}}_J),\ldots,\right). \tag{14}$$

Writing this into the two non-equalities we have that the condition of "persistent non-excitation" is the paralel fulfillment of the two non-equalities:

$$\begin{aligned}0\neq\hat{G}\Big(&(\mathbf{a},\mathbf{b}_0,\mathbf{b}_1,\ldots,\mathbf{b}_J),(\dot{\mathbf{a}},\dot{\mathbf{b}}_0,\dot{\mathbf{b}}_1,\ldots,\dot{\mathbf{b}}_J),\ldots,0,0,\ldots,\\&\hat{f}\left((\mathbf{a},\mathbf{b}_0,\mathbf{b}_1,\ldots,\mathbf{b}_J),(\dot{\mathbf{a}},\dot{\mathbf{b}}_0,\dot{\mathbf{b}}_1,\ldots,\dot{\mathbf{b}}_J),\ldots\right)\\0\neq\ &\hat{f}(\mathbf{a},\mathbf{b}_0,\mathbf{b}_1,\ldots,\mathbf{b}_J),(\dot{\mathbf{a}},\dot{\mathbf{b}}_0,\dot{\mathbf{b}}_1,\ldots,\dot{\mathbf{b}}_J)\ldots\Big)\end{aligned}$$

Negation of these statements gives the condition for persistent excitation

$$0 = \hat{G}(\mathbf{a},\mathbf{b}_0,\mathbf{b}_1,\ldots,\mathbf{b}_J),(\dot{\mathbf{a}},\dot{\mathbf{b}}_0,\dot{\mathbf{b}}_1,\ldots,\dot{\mathbf{b}}_J),\ldots,0,0,\ldots,$$
$$\hat{f}\left((\mathbf{a},\mathbf{b}_0,\mathbf{b}_1,\ldots,\mathbf{b}_J),(\dot{\mathbf{a}},\dot{\mathbf{b}}_0,\dot{\mathbf{b}}_1,\ldots,\dot{\mathbf{b}}_J),\ldots\right)$$

or

$$0 = \hat{f}\Big((\mathbf{a},\mathbf{b}_0,\mathbf{b}_1,\ldots,\mathbf{b}_J),(\dot{\mathbf{a}},\dot{\mathbf{b}}_0,\dot{\mathbf{b}}_1,\ldots,\dot{\mathbf{b}}_J)\ldots\Big).$$

Returning to the solvability of the implicit function of ξ we can obtain (14).

Again Diop's elimination theorem (algorithm) can be applied. Regard the vector ξ as a state that can be eliminated. For this we would need a state equation, a dynamics as a differential equation for ξ. But since ξ is a constant the dynamics is simply $\dot{\boldsymbol{\xi}} = 0$.

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